

Short range asymptotic behavior of the wave-functions of interacting spin-half fermionic atoms with spin-orbit coupling: a model study

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We consider spin-half fermionic atoms with isotropic Rashba spin-orbit coupling in three directions. The interatomic potential is modeled by a square well potential. We derive the analytic form of the asymptotic wave-functions at short range of two fermions in the subspace of zero net momentum and zero total angular momentum. We show that the spin-orbit coupling has perturbative effects on the short range asymptotic behavior of the wave-functions away from resonances. We argue that our conclusion should hold generally.

I. INTRODUCTION

The study of dilute unitary atomic Fermi gases has significantly and substantially extended our knowledge on strongly interacting many-body systems [1]. Theoretic advances in this direction have been propelled by observing the short-range asymptotic behavior of the wave-function of the systems [2–4]. In such ultra-cold dilute gases, when the separation r between two fermions of different species is much smaller than the mean inter-particle spacing d , but bigger than r_0 , the range of the interatomic potential, the pair wave-function of the two fermions is mainly s -wave and has the asymptotic form $\psi_s(r) = 1 - a_s/r$; when a similar situation happens to two fermions of same species, the pair wave-function is mainly p -wave and its radial part has the asymptotic form $\psi_p(r) = k(r - u_p/r^2)$. Here k is the relative wave vector between the two fermions and of order $1/d$, a_s is the s -wave scattering length and u_p is the p -wave scattering volume. Thus, when considering the short-range correlations, the p -wave part can be safely neglected compared to the s -wave part unless u_p becomes divergent. Based on this observation, a collection of remarkable relations regarding the short-range correlations and various physical observables have been derived [2–4].

On the other hand, recent successful engineering of synthetic gauge field adds another important dimension to atomic gases [5–10]. Raman processes couple different hyper-fine states of atoms and realizes a model with an effective spin-orbit coupling for spin-half particles when the hyper-fine states other than the lowest two are adiabatically eliminated. This success raises the prospect of using ultra-cold atomic systems to study and simulate spin-orbit coupling physics [11], which has excited great interest in condensed matter physics [12, 13].

The introduction of the spin-orbit coupling to ultra-cold atomic gases also raises another interesting question: How would the short-range asymptotic behavior of the wave-function of atomic gases be modified? Attempts to answer this question include: Cui studied the

two-body problem of fermions with symmetric Rashba spin-orbit coupling in three directions and focused on properties of the bound states. The discovery that the usual s -wave pseudo-potential is still applicable in certain regimes indicates that the effects of the spin-orbit coupling on the short-range asymptotic behavior of the wave-function can be perturbative [14]. Later Ref. [15] presented a general argument that since the spin-orbit coupling strength, as usually realized in the experiments, is much smaller than the energy scales associated with interatomic potentials, the wave-function inside the range of the interatomic potential should remain intact to zero order. The asymptotic behavior of the wave-function outside the interatomic potential range which is determined by the one inside the range should stay unchanged as well. Reference [16] reached a similar conclusion by an argument in which unitary transformations are implemented to eliminate the spin-orbit coupling from the kinetic part of the Hamiltonian. However, besides the case with Rashba spin-orbit coupling in one direction [16], the general arguments mentioned above await justification from explicitly worked out examples.

In this paper, we study the problem of two spin-half fermions with isotropic spin-orbit coupling in three directions. The single-particle Hamiltonian of the fermions is

$$H_1 = \frac{\mathbf{p}^2}{2m} + \frac{\lambda}{m} \mathbf{p} \cdot \boldsymbol{\sigma} + \frac{\lambda^2}{2m} \quad (1)$$

with m the mass of the atoms. Without loss of generality, we assume the spin-orbit coupling strength to be positive, $\lambda > 0$. We model the interatomic potential by an attractive square well potential, $V(r) = -V_0\theta(r_0 - r)$, where r_0 is the potential range and $V_0 > 0$. As of experimental interest, we assume $\lambda r_0 \ll 1$. We calculate the scattering and bound state wave-functions of the two interacting fermions in the subspace of zero net momentum and zero total angular momentum (sum of the spin and orbital angular momentum). We derive analytically the asymptotic behavior of the wave-functions outside the range of the square well potential. Away from resonances, the modification of the asymptotic behavior due to the spin-orbit coupling is shown to be perturbative. We explain this perturbative effect by inspecting the wave-functions

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inside the potential range and argue that our conclusion is valid for generic situations with spin-orbit coupling.

II. THE SCHRÖDINGER EQUATION

The Hamiltonian of two interacting fermions can be cast into the form

$$H_2 = H_K + H_k \quad (2)$$

$$H_K = \frac{\mathbf{K}^2}{4m} + \frac{\lambda \mathbf{K}}{2m} \cdot (\boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_1) \quad (3)$$

$$H_k = \frac{\mathbf{k}^2}{m} + \frac{\lambda \mathbf{k}}{m} \cdot (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1) + \frac{\lambda^2}{m} + V(r). \quad (4)$$

Here \mathbf{K} is the net momentum of the two fermions and \mathbf{k} is the relative one, and \mathbf{r} are the relative coordinates. The subscript of $\boldsymbol{\sigma}$ labels for the i th-fermion. From Eqs. (3) and (4), the motion of the center of mass and the relative one are coupled together via the spin operators. In the following, we focus on the subspace of $\mathbf{K} = 0$ and $\mathbf{J} \equiv \mathbf{L} + \mathbf{S} = 0$, with \mathbf{L} the orbital angular momentum and \mathbf{S} the total spin, where analytic results can be derived.

The relative wave-function $\Psi(\mathbf{r})$ in the subspace $\mathbf{K} = 0$ satisfies the Schrödinger equation

$$\left[\frac{\hat{\mathbf{k}}^2 + \lambda^2}{m} + M_\lambda + V(r) \right] \Psi(\mathbf{r}) = E \Psi(\mathbf{r}), \quad (5)$$

with

$$M_\lambda = 2 \frac{\lambda}{m} \begin{bmatrix} 0 & \frac{\hat{k}_x + i\hat{k}_y}{\sqrt{2}} & -\frac{\hat{k}_x - i\hat{k}_y}{\sqrt{2}} & -\hat{k}_z \\ \frac{\hat{k}_x - i\hat{k}_y}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{\hat{k}_x + i\hat{k}_y}{\sqrt{2}} & 0 & 0 & 0 \\ -\hat{k}_z & 0 & 0 & 0 \end{bmatrix} \quad (6)$$

when represented in the spin basis $[(\uparrow\downarrow - \downarrow\uparrow)/\sqrt{2}, \uparrow\uparrow, \downarrow\downarrow, (\uparrow\downarrow + \downarrow\uparrow)/\sqrt{2}]$ for the two spin-half fermions. Since each elements of M_λ are proportional to the spherical harmonic functions $Y_{1,m}(\Omega_{\mathbf{k}})$ or their complex conjugates, with further constraint $\mathbf{J} = 0$ we have

$$\Psi(\mathbf{r}) = \psi_0(r) \begin{bmatrix} Y_{0,0}(\Omega_{\mathbf{r}}) \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{i}{\sqrt{3}} \psi_1(r) \begin{bmatrix} 0 \\ Y_{1,-1}(\Omega_{\mathbf{r}}) \\ Y_{1,1}(\Omega_{\mathbf{r}}) \\ -Y_{1,0}(\Omega_{\mathbf{r}}) \end{bmatrix}. \quad (7)$$

The new spinor wave-function $\Phi(r) = [\psi_0(r), \psi_1(r)]^T$ satisfies

$$\left\{ -\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \lambda^2 + m[V(r) - E] \right. \\ \left. + \begin{bmatrix} 0 & -2\lambda(d/dr + 2/r) \\ 2\lambda(d/dr) & 2/r^2 \end{bmatrix} \right\} \Phi(r) = 0. \quad (8)$$

III. SCATTERING STATES

It is straightforward to show that for the scattering states, the wavefunctions are

$$\Phi^<(r) = A \begin{bmatrix} j_0(q_1 r) \\ j_1(q_1 r) \end{bmatrix} + B \begin{bmatrix} j_0(q_2 r) \\ j_1(q_2 r) \end{bmatrix} \quad (9)$$

for $r < r_0$, and

$$\Phi^>(r) = C \begin{bmatrix} h_0^{(2)}(k_1 r) \\ -h_1^{(2)}(k_1 r) \end{bmatrix} + D \begin{bmatrix} h_0^{(1)}(k_1 r) \\ -h_1^{(1)}(k_1 r) \end{bmatrix} \\ + E \begin{bmatrix} h_0^{(1)}(k_2 r) \\ h_1^{(1)}(k_2 r) \end{bmatrix} + F \begin{bmatrix} h_0^{(2)}(k_2 r) \\ h_1^{(2)}(k_2 r) \end{bmatrix} \quad (10)$$

for $r > r_0$. Here j_i are the spherical Bessel functions and $h_i^{(1,2)}$ are the spherical Hankel functions, and

$$k_1 \equiv k - \lambda, \quad k_2 \equiv k + \lambda, \quad (11)$$

$$q_1 \equiv \lambda - q, \quad q_2 \equiv \lambda + q, \quad (12)$$

$$k \equiv \sqrt{mE}, \quad q \equiv \sqrt{m(E + V_0)}. \quad (13)$$

Since there are six coefficients (A to F) and four connection conditions at $r = r_0$ for the wavefunctions and an overall normalization factor, the wavefunctions can be determined up to a coefficient; there are two linearly independent solutions for each energy E . Let us choose the i th linearly independent solution Φ_i as $\Phi_i^< = [j_0(q_i r), j_1(q_i r)]^T$ for $r < r_0$. By requiring Φ_i and its first derivative continuous at $r = r_0$, we find the corresponding coefficients for $\Phi_i^>$

$$C_i = \frac{k_2 - q_i}{k_1 + k_2} \frac{h_0^{(1)}(\tilde{k}_1) j_1(\tilde{q}_i) + h_1^{(1)}(\tilde{k}_1) j_0(\tilde{q}_i)}{h_1^{(1)}(\tilde{k}_1) h_0^{(2)}(\tilde{k}_1) - h_0^{(1)}(\tilde{k}_1) h_1^{(2)}(\tilde{k}_1)} \quad (14)$$

$$E_i = \frac{k_1 + q_i}{k_1 + k_2} \frac{h_0^{(2)}(\tilde{k}_2) j_1(\tilde{q}_i) - h_1^{(2)}(\tilde{k}_2) j_0(\tilde{q}_i)}{h_0^{(2)}(\tilde{k}_2) h_1^{(1)}(\tilde{k}_2) - h_1^{(2)}(\tilde{k}_2) h_0^{(1)}(\tilde{k}_2)} \quad (15)$$

$$D_i = C_i^* \quad (16)$$

$$F_i = E_i^*, \quad (17)$$

with $\tilde{k}_i \equiv k_i r_0$ and $\tilde{q}_i \equiv q_i r_0$.

The physical meaning of the above coefficients can be understood in the following way. Without interaction, $V(r) = 0$, either $[j_0(k_1 r), j_1(k_1 r)]^T$ or $[j_0(k_2 r), j_1(k_2 r)]^T$ is the free particle solution to Eq. (8). The interaction $V(r)$ realizes mutual scattering between the two waves. We can construct two new linearly independent solutions

as

$$\Phi_{k_1}^> = \frac{E_2^* \Phi_1^> - E_1^* \Phi_2^>}{E_2^* C_1 - E_1^* C_2} = \begin{bmatrix} h_0^{(2)}(k_1 r) \\ -h_1^{(2)}(k_1 r) \end{bmatrix} + S_{11} \begin{bmatrix} h_0^{(1)}(k_1 r) \\ -h_1^{(1)}(k_1 r) \end{bmatrix} + S_{21} \frac{k_2}{k_1} \begin{bmatrix} h_0^{(1)}(k_2 r) \\ h_1^{(1)}(k_2 r) \end{bmatrix} \quad (18)$$

$$\Phi_{k_2}^> = \frac{C_2 \Phi_1^> - C_1 \Phi_2^>}{C_2 E_1^* - C_1 E_2^*} = \begin{bmatrix} h_0^{(2)}(k_2 r) \\ h_1^{(2)}(k_2 r) \end{bmatrix} + S_{22} \begin{bmatrix} h_0^{(1)}(k_2 r) \\ h_1^{(1)}(k_2 r) \end{bmatrix} + S_{12} \frac{k_1}{k_2} \begin{bmatrix} h_0^{(1)}(k_1 r) \\ -h_1^{(1)}(k_1 r) \end{bmatrix}, \quad (19)$$

where S_{ij} are the elements of the matrix

$$S = \frac{1}{E_2^* C_1 - E_1^* C_2} \times \begin{bmatrix} E_2^* C_1^* - E_1^* C_2^* & \frac{k_2}{k_1} (C_2^* C_1 - C_1^* C_2) \\ \frac{k_1}{k_2} (E_2^* E_1 - E_1^* E_2) & C_1 E_2 - C_2 E_1 \end{bmatrix}. \quad (20)$$

Since $h_i^{(1)}(x) \sim e^{ix}/x$ and $h_i^{(2)}(x) \sim e^{-ix}/x$ when $x \rightarrow \infty$, the wave-function $\Phi_{k_1}^> (\Phi_{k_2}^>)$ describes the process that the incoming wave of wave vector $k_1 (k_2)$ is scattered by $V(r)$ into the outgoing waves of wave vectors k_1 and k_2 . The matrix S formed by the coefficients S_{ij} is the scattering S -matrix. It is straightforward to show that S satisfies the unitary condition $S^\dagger S = 1$. The appearance of the ratios between k_1 and k_2 in Eq. (20) is because the two wave vectors are of different magnitude.

We can further obtain the wave-functions such that the S -matrix is diagonalized:

$$\begin{aligned} \Phi_\pm^> &= v_1^\pm k_1 \Phi_{k_1}^> + v_2^\pm k_2 \Phi_{k_2}^> \\ &= v_1^\pm k_1 \begin{bmatrix} h_0^{(2)}(k_1 r) \\ -h_1^{(2)}(k_1 r) \end{bmatrix} + v_2^\pm k_2 \begin{bmatrix} h_0^{(2)}(k_2 r) \\ h_1^{(2)}(k_2 r) \end{bmatrix} \\ &\quad + e^{2i\delta_\pm} \left\{ v_1^\pm k_1 \begin{bmatrix} h_0^{(1)}(k_1 r) \\ -h_1^{(1)}(k_1 r) \end{bmatrix} + v_2^\pm k_2 \begin{bmatrix} h_0^{(1)}(k_2 r) \\ h_1^{(1)}(k_2 r) \end{bmatrix} \right\}, \end{aligned} \quad (21)$$

where $[v_1^\pm, v_2^\pm]^T$ are the eigenvectors of S given by Eq. (20) and $e^{2i\delta_\pm}$ are the corresponding eigenvalues. From Eqs. (14) and (15), one can prove $k_1^2(E_2^* E_1 - E_1^* E_2) = k_2^2(C_2^* C_1 - C_1^* C_2)$. Since the S -matrix can be expanded as $S = \tilde{S}_0 \mathbb{1} + \tilde{S}_x \sigma_x + \tilde{S}_z \sigma_z$ and the ratio between the coefficients \tilde{S}_x and \tilde{S}_z is real, we can choose the eigenvectors $[v_1^\pm, v_2^\pm]^T$ to be real. The solutions $\Phi_\pm^>$ are standing waves since $[\Phi_\pm^>]^*$ is proportional to $\Phi_\pm^>$ apart from a phase. This complies with the expectation that solutions to the one-dimensional differential equation (8) can be chosen to be real. Equation (21) corresponds to the ansatz used in Ref. [14] [cf. Eqs. (31) and (32) therein]. The two new phase shifts δ_\pm are the counterparts of the s -wave and p -wave phase shifts δ_s and δ_p in the absence of the spin-orbit coupling.

We plot $\delta_-(kr_0)$ in Fig. (1) and $\delta_+(kr_0)$ in Fig. (2) for $\tilde{\lambda} = 0.01$ and $\eta = 1$ with $\tilde{\lambda} \equiv \lambda r_0$ and $\eta \equiv \sqrt{mV_0 r_0^2}$. Generically we find $\exp[2i\delta_-(0)] = -1$ and $\exp[2i\delta_+(0)] = 1$. The unitarity of $\delta_-(0)$ is a reminiscent of one-dimensional scattering [14]. When both δ_- and δ_+ are defined in the domain $[-\pi/2, \pi/2]$, to zero order of $\tilde{\lambda}$, $\delta_-(0)$ jumps by π where a_s switches between $-\infty$ to $+\infty$, and the slope of $\delta_-(k)$ in the small k limit changes sign where u_p flips between $-\infty$ to $+\infty$. To analyze the properties of δ_\pm , we expand the S -matrix to first order of k

$$S = \sigma_x + M\tilde{k} + \mathcal{O}(\tilde{k}^2), \quad (22)$$

where $\tilde{k} \equiv kr_0$. The elements M_{ij} of the matrix M are listed in the Appendix. Correspondingly in such limit, we have $\delta_- = -\text{sgn}(\delta'_-) \pi/2 + \delta'_- \tilde{k}$, $\delta_+ = \delta'_+ \tilde{k}$ and

$$\mathbf{v}^- = \frac{1}{\sqrt{2}} \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \delta v \tilde{k} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right), \quad (23)$$

$$\mathbf{v}^+ = \frac{1}{\sqrt{2}} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \delta v \tilde{k} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right), \quad (24)$$

with

$$\begin{aligned} \delta'_- &= \frac{i}{4} (M_{11} + M_{22} - M_{12} - M_{21}) \\ &= \{2\tilde{\lambda}^4 - 2\eta^2 \tilde{\lambda}^2 + \eta[\eta^2 + \tilde{\lambda}^2 - 2\eta^2 \tilde{\lambda}^2 + 2\tilde{\lambda}^4 + (\eta^2 + \tilde{\lambda}^2) \cos(2\tilde{\lambda})] \sin(2\eta) + 2\eta^2 \tilde{\lambda} [\tilde{\lambda} \cos(2\tilde{\lambda}) - \sin(2\tilde{\lambda})] \\ &\quad + 2\tilde{\lambda} \cos(2\eta) [-2\eta^2 \tilde{\lambda} + \tilde{\lambda}^3 - \eta^2 \sin(2\tilde{\lambda})]\} / \{2\eta^2 \tilde{\lambda}^2 [\cos(2\eta) - \cos(2\tilde{\lambda})] + 2\eta \tilde{\lambda}^2 (\eta^2 - \tilde{\lambda}^2) \sin(2\eta)\}, \end{aligned} \quad (25)$$

$$\begin{aligned} \delta'_+ &= -\frac{i}{4} (M_{11} + M_{22} + M_{12} + M_{21}) \\ &= \{\eta[\eta^2 + \tilde{\lambda}^2 - 2\eta^2 \tilde{\lambda}^2 + 2\tilde{\lambda}^4 - (\eta^2 + \tilde{\lambda}^2) \cos(2\tilde{\lambda})] \sin(2\eta) + 2\tilde{\lambda} \cos(2\eta) [\tilde{\lambda}^3 - 2\eta^2 \tilde{\lambda} + \eta^2 \sin(2\tilde{\lambda})] \\ &\quad - 2\tilde{\lambda} [\tilde{\lambda}^3 - \eta^2 \tilde{\lambda} - \eta^2 \tilde{\lambda} \cos(2\tilde{\lambda}) + \eta^2 \sin(2\tilde{\lambda})]\} / \{2\eta^2 \tilde{\lambda}^2 [\cos(2\eta) - \cos(2\tilde{\lambda})] + 2\eta \tilde{\lambda}^2 (\eta^2 - \tilde{\lambda}^2) \sin(2\eta)\}, \end{aligned} \quad (26)$$

$$\delta v = (M_{11} - M_{22})/4 = \{2\eta\tilde{\lambda}[\cos(2\eta)\cos(2\tilde{\lambda}) - 1] + (\eta^2 + \tilde{\lambda}^2)\sin(2\eta)\sin(2\tilde{\lambda})\} / \{2\eta\tilde{\lambda}^2[\cos(2\eta) - \cos(2\tilde{\lambda})] + 2\tilde{\lambda}^2(\eta^2 - \tilde{\lambda}^2)\sin(2\eta)\}. \quad (27)$$

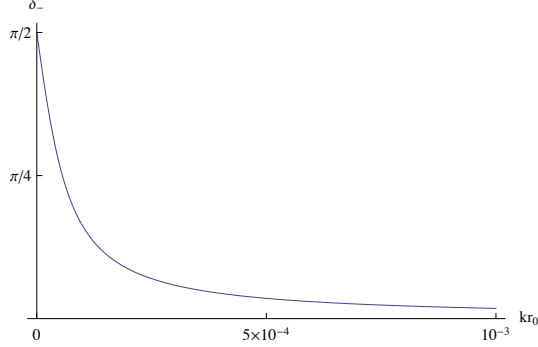


FIG. 1: Phase shift $\delta_-(kr_0)$ vs kr_0 for $\tilde{\lambda} = 0.01$ and $\eta = 1$.

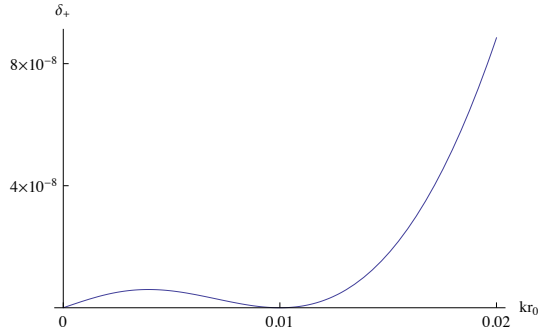


FIG. 2: Phase shift $\delta_+(kr_0)$ vs kr_0 for $\tilde{\lambda} = 0.01$ and $\eta = 1$.

The three quantities, δ'_- , δ'_+ and δv , diverge when

$$\frac{\eta}{\eta - \tilde{\lambda}}[\cos(2\eta) - \cos(2\tilde{\lambda})] + (\eta + \tilde{\lambda})\sin(2\eta) = 0, \quad (28)$$

where a bound state emerges [cf. Eq. (38)]. This divergence reflects that bound state energies are the poles of the S -matrix. Away from the singularities, these quantities have the asymptotic expansions in $\tilde{\lambda}$ as

$$\delta'_- = \frac{r_0}{\tilde{\lambda}^2 a_s} \left[1 + \mathcal{O}(\tilde{\lambda}^2) \right], \quad (29)$$

$$\delta'_+ = -\frac{\lambda^2 u_p}{3r_0} \left[1 + \mathcal{O}(\tilde{\lambda}^2) \right], \quad (30)$$

$$\delta v = \left[\frac{1}{\tilde{\lambda}} - \frac{2\lambda u_p}{3a_s r_0} \right] + \mathcal{O}(\tilde{\lambda}^3). \quad (31)$$

For our square well potential model, the s -wave scattering length a_s is given by $r_0/a_s = \eta/(\eta - \tan \eta)$, and the p -wave scattering volume u_p by $u_p/r_0^3 = 1 - 3(1 - \eta \cot \eta)/\eta^2$. Equations (29) and (30) explain the jumps of $\delta_-(0)$ and the sign change of the slope of $\delta_+(k)$ for small k .

Exceptions occur at resonances: at the first set of resonances given by Eq. (41), we find $\delta_-(0) = -\delta_+(0) \approx -\tilde{\lambda}/2$, $\mathbf{v}^- = [0, 1]^T$, and $\mathbf{v}^+ = [1, 0]^T$; at the second set, $\delta_-(0) = -\delta_+(0) \approx 2\tilde{\lambda}^3/3$, $\mathbf{v}^- = [0, 1]^T$, and $\mathbf{v}^+ = [1, 0]^T$.

The asymptotic form of the scattering wave-functions in the region $r \gtrsim r_0$ is

$$\Phi_{\pm}^{\pm} = \left\{ \frac{v_1^{\pm} + v_2^{\pm}}{r} + (v_1^{\pm} k_1 + v_2^{\pm} k_2) \cot \delta_{\pm} \right\} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \left\{ \left(\frac{v_2^{\pm}}{k_2} - \frac{v_1^{\pm}}{k_1} \right) \frac{1}{r^2} + \frac{1}{2} (v_2^{\pm} k_2 - v_1^{\pm} k_1) + \frac{1}{3} \cot \delta_{\pm} (k_2^2 v_2^{\pm} - k_1^2 v_1^{\pm}) r \right\} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (32)$$

Note that an overall factor $2e^{\delta_{\pm}} \sin \delta_{\pm}$ has been omitted. In the low energy limit $k \rightarrow 0$, when away from resonances, using Eqs. (29) to (31), we find

$$\Phi_-^{\pm} \sim \left(\frac{1}{r} - \frac{1}{a_s} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda \left(-\frac{2u_p}{3a_s r^2} + 1 + \frac{r}{3a_s} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (33)$$

$$\Phi_+^{\pm} \sim \left(\frac{1}{r} - \frac{2}{a_s} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \left(\frac{1}{\lambda r^2} + \frac{\lambda}{2} - \frac{r}{\lambda u_p} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (34)$$

At the first set of resonances given by Eq. (41),

$$\Phi_-^{\pm} = \Phi_+^{\pm} \sim \begin{bmatrix} 1/r - 2/r_0 \\ 1/\lambda r^2 + \lambda/2 - 2\lambda r/3r_0 \end{bmatrix}; \quad (35)$$

at the second set

$$\Phi_-^{\pm} = \Phi_+^{\pm} \sim \begin{bmatrix} 1/r + 3/2\lambda^2 r_0^3 \\ 1/\lambda r^2 + \lambda/2 + r/2\lambda r_0^3 \end{bmatrix}, \quad (36)$$

Note that Eqs. (33) to (36) satisfy Eq. (8) with $E = 0$ to the leading order of λ .

IV. BOUND STATES

The bound state solutions to Eq. (8) with energy E are Eq. (9) for $r < r_0$ and

$$\Phi^>(r) = C' \begin{bmatrix} h_0^{(1)}(k'_1 r) \\ h_1^{(1)}(k'_1 r) \end{bmatrix} + D' \begin{bmatrix} h_0^{(2)}(k'_2 r) \\ h_1^{(2)}(k'_2 r) \end{bmatrix} \quad (37)$$

for $r > r_0$. Here $k'_1 \equiv \lambda + i\kappa$, $k'_2 \equiv \lambda - i\kappa$, and $\kappa \equiv \sqrt{-mE}$. The requirement that the wave-function and its first derivative are continuous at $r = r_0$ gives rise to the equation determining the energy E :

$$0 = (\tilde{\lambda}^2 + \tilde{\kappa}^2 + \tilde{\kappa})(\tilde{q}^2 - \tilde{\kappa}^2)[j_0(\tilde{q}_1)j_1(\tilde{q}_2) - j_0(\tilde{q}_2)j_1(\tilde{q}_1)] + 2\tilde{\lambda}\tilde{\kappa}\tilde{q}[j_0(\tilde{q}_2)j_1(\tilde{q}_1) + j_0(\tilde{q}_1)j_1(\tilde{q}_2)] - 2\tilde{\kappa}\tilde{q}[(\tilde{\kappa} + 1)^2 + \tilde{\lambda}^2]j_0(\tilde{q}_1)j_0(\tilde{q}_2) - 2\tilde{\kappa}\tilde{q}(\tilde{\lambda}^2 + \tilde{\kappa}^2)j_1(\tilde{q}_1)j_1(\tilde{q}_2), \quad (38)$$

with $\tilde{\kappa} \equiv \kappa r_0$ and $\tilde{q} \equiv q r_0$. Assuming $D' = 1$, we have

$$C' \equiv e^{2i\theta} = -\frac{2\tilde{q}h_1^{(2)}(\tilde{k}_2)j_0(\tilde{q}_1)j_0(\tilde{q}_2) - h_0^{(2)}(\tilde{k}_2)[j_0(\tilde{q}_1)j_1(\tilde{q}_2)(\tilde{q} - i\tilde{\kappa}) + j_0(\tilde{q}_2)j_1(\tilde{q}_1)(\tilde{q} + i\tilde{\kappa})]}{2\tilde{q}h_1^{(1)}(\tilde{k}_1)j_0(\tilde{q}_1)j_0(\tilde{q}_2) - h_0^{(1)}(\tilde{k}_1)[j_0(\tilde{q}_1)j_1(\tilde{q}_2)(\tilde{q} + i\tilde{\kappa}) + j_0(\tilde{q}_2)j_1(\tilde{q}_1)(\tilde{q} - i\tilde{\kappa})]}. \quad (39)$$

In the weak attraction limit $\eta \rightarrow 0$, we find

$$\tilde{\kappa} = \eta^2 \left[1 - \left(\frac{\sin \tilde{\lambda}}{\tilde{\lambda}} \right)^2 \right]; \quad (40)$$

there is always a bound state no matter how weak the attraction is. This bound state disappears in the absence of the spin-orbit coupling since from Eq. (40) $\tilde{\kappa} \rightarrow 0$ for $\tilde{\lambda} \rightarrow 0$. The energies of bound states other than in the weak attraction limit have been studied in Ref. [14].

According to Eq. (38), a new bound state emerges at threshold $E = 0$ when Eq. (28) is satisfied. In the limit $\tilde{\lambda} \rightarrow 0$, Eq. (28) reduces to $j_0(\eta) = 0$ or $j_1(\eta) = 0$. In the absence of the spin-orbit coupling, the former condition is where a bound state emerges in the p -wave channel, while the latter is where the s -wave scattering length a_s is zero. The introduction of the spin-orbit coupling has the nonperturbative effect of shifting the onset of a series of bound states from where $1/a_s = 0$ to where $a_s = 0$. For small $\tilde{\lambda}$, the critical values of the depth of square well potential V_c where a bound state forms are given by

$$\eta_c \equiv \sqrt{mV_c r_0^2} = \begin{cases} m\pi - \tilde{\lambda}^2/m\pi + \mathcal{O}(\tilde{\lambda}^4), \\ \ell_n + \tilde{\lambda}^2/\ell_n + \mathcal{O}(\tilde{\lambda}^4), \end{cases} \quad (41)$$

with $m = 1, 2, 3, \dots$ and nonzero ℓ_n ($n = 1, 2, 3, \dots$) being the solutions of $j_1(\ell_n) = 0$. The change of V_c due to small λ agrees with the numerical results obtained in Ref. [14].

To obtain the asymptotic behavior of the wave-function of the most shallow bound state for $r \gtrsim r_0$ when V_0 is close to a critical value V_c , let us first look at the limit $\eta \rightarrow \eta_c = 0$ where there is only one bound state. From Eq. (40), in the limit $\tilde{\lambda} \rightarrow 0$, for the bound state, we have $\tilde{\kappa} = \eta^2 \tilde{\lambda}^2/3$. Substituting this into Eq. (39) and expanding in the order $0 < \tilde{\kappa} \ll \eta \ll \tilde{\lambda} \ll 1$, we obtain

$$\theta = 2 \arctan \left(\frac{\tilde{\kappa}}{\tilde{\lambda}} \right) [1 + \mathcal{O}(\tilde{\lambda}^2)] \simeq 2 \frac{\tilde{\kappa}}{\tilde{\lambda}}, \quad (42)$$

and thus for $r \gtrsim r_0$ the wave-function of the bound state asymptotically is

$$\Phi^>(r) \sim \left[\frac{1/r - 1/a_s}{-2\lambda r_0^2/15r^2 + \lambda} \right], \quad (43)$$

which coincides with the $\eta \rightarrow 0$ limit of Eq. (33).

When V_0 approaches one of the first set of critical values of Eq. (41),

$$\theta \simeq -\frac{1}{2}\tilde{\lambda}, \quad (44)$$

and

$$\Phi^>(r) \sim \left[\frac{1/r - 2/r_0}{1/\lambda r^2 + \lambda/2} \right], \quad (45)$$

which agrees with Eq. (35) to the corresponding order of r . For V_0 close to the second set of critical values of Eq. (41),

$$\theta \simeq \frac{2}{3}\tilde{\lambda}^3 \quad (46)$$

with the asymptotic wave-function

$$\Phi^>(r) \sim \left[\frac{1/r + 3/2\lambda^2 r_0^3}{1/\lambda r^2 + \lambda/2} \right], \quad (47)$$

which is the same as Eq. (36) to the corresponding order of r .

V. DISCUSSION

Equations (33) and (34) indicate that away from resonances the modification of the asymptotic forms of the wave-functions at short range due to the spin-orbit coupling is perturbative. To see this point clearly, we multiply Eq. (34) by λ and have

$$\Phi_-^> \sim \left(\frac{1}{r} - \frac{1}{a_s} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda \left(-\frac{2u_p}{3a_s r^2} + 1 + \frac{r}{3a_s} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (48)$$

$$\Phi_+^> \sim \left(\frac{1}{r^2} - \frac{r}{u_p} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \lambda \left(\frac{1}{r} - \frac{2}{a_s} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad (49)$$

apart from certain factors, to lowest order of λ , $\Phi_-^>$ have the same asymptotic form as the s -wave-function ψ_s and $\Phi_+^>$ as the p -wave-function ψ_p . The perturbative modification can be understood by inspecting the wave-functions inside the range of the potential. For $r < r_0$, in the limit $k \rightarrow 0$ and expanded to the first order of λ ,

$$\Phi_-^<(r) = \Phi_1^<(r) + \left[1 + 4 \left(\frac{1}{\eta} - \cot \eta \right) \tilde{\lambda} \right] \Phi_2^<(r), \quad (50)$$

and

$$\begin{aligned} \Phi_+^<(r) \\ = \Phi_1^<(r) - \left[1 + \frac{4}{3} \left(\frac{3 - \eta^2}{\eta} + \frac{\eta^2}{\eta - \tan \eta} \right) \tilde{\lambda} \right] \Phi_2^<(r). \end{aligned} \quad (51)$$

Thus to zero order of $\tilde{\lambda}$, $\Phi_+^<(r)$ and $\Phi_-^<(r)$ are the same as the s - and p -waves inside the square well potential in the absence of the spin-orbit coupling respectively. Since the coefficients of the terms linear in λ are well-behaved away from the resonances, nonzero $\tilde{\lambda}$ adds corrections perturbatively. Since $\Phi^>(r)$ and $\Phi^<(r)$ need to connect at $r = r_0$, the change of the asymptotic form of $\Phi^>(r)$ is bound to be perturbative in $\tilde{\lambda}$ as well.

The above analysis is essentially the same as the previous argument based on the comparison of energy scales given in Ref. [15]: Within the potential range, the spin-orbit coupling strength is much smaller than $1/r_0$ and $\sqrt{mV_0}$ and therefore generally has perturbative effects on the wave-functions in this regime, and so does on the wave-functions in the regime $r \gtrsim r_0$. According to this general consideration, we expect the same conclusion applies to the cases with nonzero pair net momentum \mathbf{K} and $\mathbf{J} \neq 0$, and to the ones with generic forms of spin-orbit coupling, given additionally $K \ll 1/r_0, \sqrt{mV_0}$. Since Ref. [16] assumes implicitly $u_p = 0$, the result derived there agrees with Eq. (48) while there is no match for Eq. (49).

Nevertheless, Eqs. (50) and (51) signal that the perturbation expansions in $\tilde{\lambda}$ break down when $a_s \rightarrow 0$ ($\eta - \tan \eta = 0$) and $1/u_p \rightarrow 0$ ($\tan \eta = 0$ for $\eta \neq 0$), which are the conditions for the emergence of a new bound state in the limit $\tilde{\lambda} \rightarrow 0$. At the first set of resonances, the asymptotic forms (35) suggests that $\tilde{\lambda}$ can still be treated

perturbatively. At the second set of resonances, extra powers of λ in the denominators of (36) become singular as $\lambda \rightarrow 0$ there. This coincides with the perturbative and nonperturbative shifts of the onset of bound states compared to the cases without the spin-orbit coupling mentioned in Sec. IV.

Our calculations suggest that for generic spin-orbit coupling, no matter for bosons or fermions, the corrections to the asymptotic form of the wave-functions at short range should be perturbative except for at fine tuned points where resonances occur. Consequently the relations derived based on the asymptotic form [2–4] should stand valid correspondingly.

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Appendix A: Elements of the M matrix

We list the elements of the M matrix here:

$$M_{11} = \frac{e^{-2i(\eta-\tilde{\lambda})}\eta(\eta+\tilde{\lambda})^2 - e^{2i(\eta+\tilde{\lambda})}\eta(\eta-\tilde{\lambda})^2 + 4i\tilde{\lambda}[\eta^2(i+\tilde{\lambda}) - \tilde{\lambda}^3]}{2\eta\tilde{\lambda}^2\{\eta[\cos(2\eta) - \cos(2\tilde{\lambda})] + (\eta^2 - \tilde{\lambda}^2)\sin(2\eta)\}}, \quad (A1)$$

$$M_{12} = -i \frac{2\tilde{\lambda}^2(2\eta^2 - \tilde{\lambda}^2)\cos(2\eta) - 2\eta^2\tilde{\lambda}^2\cos(2\tilde{\lambda}) + \eta[(2\eta^2 - 1)\tilde{\lambda}^2 - 2\tilde{\lambda}^4 - \eta^2]\sin(2\eta) + 2\eta^2\tilde{\lambda}\sin(2\tilde{\lambda})}{\eta\tilde{\lambda}^2\{\eta[\cos(2\eta) - \cos(2\tilde{\lambda})] + (\eta^2 - \tilde{\lambda}^2)\sin(2\eta)\}}, \quad (A2)$$

$$M_{21} = M_{12}, \quad (A3)$$

$$M_{22} = -M_{11}^*. \quad (A4)$$

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